

Some integrals occurring in a topology change problem

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(February 7, 2000)

In a paper presented a few years ago, De Lorenci *et al.* showed, in the context of canonical quantum cosmology, a model which allowed space topology changes. The purpose of this present work is to go a step further in that model, by performing some calculations only estimated there for several compact manifolds of constant negative curvature, such as the Weeks and Thurston spaces and the icosahedral hyperbolic space (Best space).

PACS numbers: 98.80.Hw, 02.40.-k

I. INTRODUCTION

A few years ago De Lorenci *et al.* [1] presented a model of quantum cosmology which allowed space topology changes, having as main idea the use of the “conditional probability interpretation” to establish selection rules for the possible changes of topology; the wavefunctions involved in the process were of the type

$$\Psi = \Psi(\alpha, \beta, \xi, \phi) = A_k(a, \phi) e^{\xi F_k}, \quad (1)$$

where α and β are appropriated canonical variables built upon the more common set of spherical coordinates (χ, θ, φ) , the scale factor a and the curvature k ; ξ and ϕ are, respectively, a dust field describing a “distribution of irrotational dust particles” and a scalar field, both representing the matter content of the model; and F_k is basically a numerical coefficient obtained by integration of certain functions constructed upon the ‘value’ $\chi_0(\theta, \varphi; V^3)$ of the radial coordinate of the fundamental polyhedron’s boundary of the 3-dimensional manifold V^3 , of curvature k , considered, written explicitly as

$$F_k = \frac{a}{2\pi\hbar m} \int_{V^3} \frac{\sin 2\sqrt{k}\chi_0(\theta, \varphi; V^3)}{2\sqrt{k}} d\theta \sin\theta d\varphi. \quad (2)$$

The topology changes would occur at some value ξ of the dust field, when $a = \bar{a}$ and $\phi = \bar{\phi}$, such that the conditional probability of having $k = -1, 0$ or $+1$ would be

$$\begin{aligned} P_c(k|\bar{a}, \bar{\phi}) &= \frac{|\Psi(k, \bar{a}, \bar{\phi})|^2}{\sum_{k=0, \pm 1} |\Psi(k, \bar{a}, \bar{\phi})|^2} \\ &= \frac{A_k^2(\bar{a}, \bar{\phi}) e^{2\xi F_k}}{\sum_{k'=0, \pm 1} A_{k'}^2(\bar{a}, \bar{\phi}) e^{2\xi F_{k'}}}. \end{aligned} \quad (3)$$

So, when $\xi \rightarrow \pm\infty$ one has one of the $P_c(k|\bar{a}, \bar{\phi})$ equal to one and the other two null, depending upon the value of F_k .

In [1] the values of the functions F_k were only estimated for two different compact manifolds, the Poincaré

dodecahedral space D^3 , of positive curvature, and the hyperbolic icosahedral space I^3 (also known as Best space), of negative curvature; since there the authors claimed that “it is not possible to calculate the F_i ’s exactly” for these manifolds, the importance of the present work is in the exact calculation of the functions F_k for several compact manifolds of constant negative curvature, including the cited I^3 .

II. SOME CALCULUS IN COMPACT MANIFOLDS

The functions F_k , such as presented in equation (2), are probably uncomputable since the specific form of the functions $\chi_0(\theta, \varphi; V^3)$ are difficult, if not impossible, to determine; however, one can simply establish the following limits for the F_k ’s:

$$4\pi \frac{\sin 2\sqrt{k}\chi_{\min}}{2\sqrt{k}} \leq \frac{F_k}{(a/2\pi\hbar m)} \leq 4\pi \frac{\sin 2\sqrt{k}\chi_{\max}}{2\sqrt{k}}, \quad (4)$$

where χ_{\min} and χ_{\max} are, respectively, the radii of the inscribed and circumscribed circumference of the the fundamental cell of the manifold in consideration. In [1] the functions χ_0 appear after performing an “integration with respect to the variable χ ”, using as interval of integration $[0, \chi_0(\theta, \varphi; V^3)]$; so, in order to obtain a numerical value for the functions F_k , it is easy to see that one can start with the integral

$$F_k = -\frac{a}{2\pi\hbar m} \int_{V^3} [\sin^2 \sqrt{k}\chi - \cos^2 \sqrt{k}\chi] d\chi d\theta \sin\theta d\varphi. \quad (5)$$

Noticing now that

$$dV = \frac{\sin^2 \sqrt{k}\chi}{k} d\chi d\theta \sin\theta d\varphi \quad (6)$$

is simply the element of volume for the spatial part of a Friedmann-Robertson-Walker metric, written in spherical coordinates, there are two possible ways to follow,

one plainer and the other a little more sophisticated; in both, however, one needs to redefine the coordinates and limits of integration used. So, the next step consists in the use of cylindrical coordinates (ρ, φ, z) , related to the spherical coordinates (χ, θ, φ) by means of the relations

$$\begin{cases} \cos \sqrt{k}\chi = \cos \sqrt{k}\rho \cos \sqrt{k}z \\ \sin \sqrt{k}\chi \sin \theta = \sin \sqrt{k}\rho \end{cases} \quad (7)$$

or

$$d\chi^2 + \frac{\sin^2 \sqrt{k}\chi}{k} [d\theta^2 + \sin^2 \theta d\varphi^2] = d\rho^2 + \cos^2 \sqrt{k}\rho dz^2 + \frac{\sin^2 \sqrt{k}\rho}{k} d\varphi^2. \quad (8)$$

Now, one has the interval of integration $[0, \rho_0(z, \varphi; V^3)]$ for the coordinate ρ ; the expression for $\rho_0(z, \varphi; V^3)$ is easily obtainable, since is only a matter of using trigonometrical identities in the plane, *i.e.*, in the triangles that compose the faces of each tetrahedron in which the fundamental polyhedron can be divided* using the following procedure:

- for each face draw a geodesic line perpendicular to it, connecting it to the center A of the polyhedron, and crossing it or its plane in a point B (this line AB gives the height z of the tetrahedron);
- for each edge draw a geodesic line perpendicular to it and connecting it to the point B of the face to which the edge belongs, crossing the edge or its extension in a point C ;
- complete the tetrahedron with one of the two vertices of the edge, naming it as D .

These steps will create some ‘negative’ tetrahedra, covering also regions outside the polyhedron, and some ‘positive’, covering only regions of the polyhedron, each one of them having four right-angled triangles, one of which (named here BCD) is the base of the tetrahedron; integration on the compact manifold represented by the polyhedron is the difference between the sums of the integrations on all of the positive tetrahedra and the integrations on all of the negative tetrahedra.

*For more information on trigonometric identities in non-euclidean spaces one can see references [2] to [6]; [3] is a classical book of cosmology with one section on spherical trigonometry.

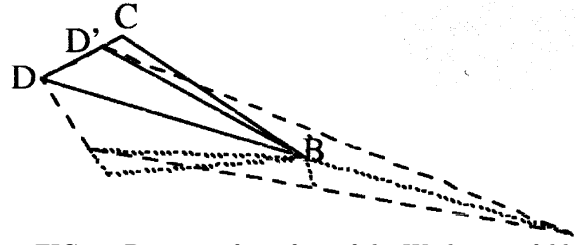


FIG. 1. Division of one face of the Weeks manifold (quadrilateral area bounded by the segmented lines) in several triangles; the triangles labeled BCD and BCD' form, when connected to the center A of the fundamental polyhedron, the basis of one ‘positive’ and one ‘negative’ tetrahedron, respectively. Notice that the point A , not shown in the figure, belongs to a different plane.

The easiest path of integration consists of simply making

$$\begin{aligned} F_k &= -\frac{a}{2\pi\hbar m} \int_{V^3} [2\sin^2 \sqrt{k}\chi - 1] d\chi d\theta \sin \theta d\varphi \quad (9) \\ &= -\frac{a}{2\pi\hbar m} \left[2kv - \int_{V^3} \frac{kdV}{\sin^2 \sqrt{k}\chi} \right], \end{aligned}$$

where v is the volume of the compact manifold where the integration is being performed. The remaining integral in the right hand side of the last equality must be done in the new set of cylindrical coordinates, where the limits of integration for the particular case of negative curvature ($k = -1$) are, in each tetrahedron[†],

$$0 \leq z \leq d_{AB}, \quad 0 \leq \varphi \leq \hat{C}BD, \quad (10)$$

and

$$0 \leq \rho \leq \rho_0(z, \varphi) = \operatorname{arctanh} \left[\tan \hat{B}AC \frac{\sinh z}{\cos \varphi} \right]. \quad (11)$$

The integration in the coordinate ρ is easily done and gives finally

$$\begin{aligned} \frac{F_{-1}}{a/2\pi\hbar m} &= 2v + \int_0^{\hat{C}BD} d\varphi \int_0^{d_{AB}} \frac{dz}{\cosh^2 z} \ln \sqrt{\frac{\cos^2 \varphi + \tan^2 \hat{B}AC}{\cos^2 \varphi - \tan^2 \hat{B}AC} \sinh^2 z} \end{aligned} \quad (12)$$

from where numerical results can be obtained by plain numerical integration. Notice that the same procedure can yield a formula for the volume of the manifold.

Alternatively, one can start doing

[†]The trigonometric identities that lead to such result are showed in an appendix at the end of this work.

$$F_k = -\frac{ak}{2\pi\hbar m} \int_{V^3} [1 - \cot^2 \sqrt{k}\chi] dV \quad (13)$$

$$= -\frac{ak}{2\pi\hbar m} \left[v + \int_{V^3} \nabla_\mu V^\mu dV \right]$$

where V^μ is a vector satisfying the differential equation

$$\nabla_\mu V^\mu = (\partial_\mu + \Gamma_{\mu\rho}^\rho) V^\mu = -k \cot^2 \sqrt{k}\chi, \quad (14)$$

whose solution in spherical coordinates is[‡]

$$V^\chi = -\frac{1}{2} \left[\sqrt{k} \cot \sqrt{k}\chi + k\chi \csc^2 \sqrt{k}\chi \right] \quad (15)$$

$$= -\frac{1}{\chi} - \sum_{\ell=1}^{\infty} \frac{k^2 \chi^3}{(k\chi^2 - \pi^2 \ell^2)^2},$$

where the last equality was put to show clearly the behavior of the solution when $k = 0$. This result permits the use of Stokes's theorem [8] to make

$$- \int_{V^3} k \cot^2 \sqrt{k}\chi dV = \int_{S=\partial V^3} g_{\mu\nu} V^\mu n^\nu dA, \quad (16)$$

where n^ν is a vector normal to the boundary S of the fundamental cell of the compact manifold V^3 , obeying the constraint $n^\mu n_\mu = 1$.

In the procedure presented here the faces of the fundamental polyhedron that represents a compact manifold appear, by construction, as surfaces of constant z , allowing to use as element of area

$$dA = \frac{\sin \sqrt{k}\rho}{\sqrt{k}} d\rho d\varphi. \quad (17)$$

Finally, to carry out the integration the vector V^χ must be written in cylindrical coordinates; only the component $V^z = V^\chi \partial_\chi z$, normal to the base of the tetrahedron, is important. This procedure can be used also to give the volume of each tetrahedron, what allows to write, in the case of negative curvature,

$$\frac{F_{-1}}{a/2\pi\hbar m} = \quad (18)$$

$$\int_0^{\hat{C}\hat{B}\hat{D}} \frac{d\varphi}{\coth z} \ln \sqrt{\frac{\cos^2 \varphi + \tan^2 \hat{B}\hat{A}\hat{C}}{\cos^2 \varphi - \tan^2 \hat{B}\hat{A}\hat{C} \sinh^2 z}},$$

where $z = d_{AB}$.

[‡]Here is used the identity

$$\frac{\pi^2}{4m^2} \csc^2 \frac{\pi}{m} + \frac{\pi}{4m} \cot \frac{\pi}{m} - \frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{(1 - k^2 m^2)^2}$$

found as equation 1.423 of reference [7].

III. NUMERICAL RESULTS

To obtain numerical results the data – volumes and coordinates of all vertices for several hyperbolic compact manifolds – contained in the literature were used (see, for instance, [9] e [10]) together with those of the software *SnapPea*[§] [11]; part of the data used are presented in Table I. The manifolds choosed present in some way a degree of symmetry which simplified the calculus, but, in principle, the approach followed can be used to any compact manifold. All results are presented in the Table II where they are compared with estimates done as in [1]; the result obtained for the Weeks manifold was used in [12].

TABLE I. Data for each manifold studied.

Manifold	volume	χ_{min}	χ_{max}
Weeks	0.942707	0.519162	0.752470
Thurston	0.981369	0.535437	0.748538
<i>m036</i> (−3, 2)	2.029883	0.675646	1.014814
<i>m016</i> (−4, 3)	2.343017	0.691286	0.895576
<i>m036</i> (−2, 3)	2.568971	0.726205	0.895576
Best	4.686034	0.868298	1.382571
<i>v3469</i> (+3, 1)	5.137941	0.808931	1.45241

TABLE II. Summary of the results.

Manifold	$F_{-1}/(a/2\pi\hbar m)$	$2\pi \sinh 2\chi_{min}$	$2\pi \sinh 2\chi_{max}$
Weeks	9.28474	7.76109	13.4518
Thurston	9.48385	8.09029	13.3355
<i>m036</i> (−3, 2)	13.4897	11.3208	23.4987
<i>m016</i> (−4, 3)	14.5526	11.7314	18.3142
<i>m036</i> (−2, 3)	15.3167	12.6901	20.6181
Best	21.4948	17.2847	49.6976
<i>v3469</i> (+3, 1)	22.5418	15.2178	57.1996

IV. CONCLUSION

There are several formulations of quantum cosmology and the intention of this work is to put some new light over a particular one, showing that the wavefunctions built by the procedure of [1] present a dependence on the volume of the compact manifold in consideration; aside that, such wavefunctions have an additional dependence on the *shape* of the fundamental cell of the manifold, due to a surface term that does not appear in several other models.

[§] *SnapPea* is an electronic catalog of thousands of hyperbolic compact manifolds, each one of them identified by volume and a code such as *m036*(−3, 2).

To finish, it is also interesting to notice that the results presented here, though of specific relevance for a particular model of quantum cosmology, can be seen in a more generalized context, since this work presents a method that allows to easily calculate the volume of the fundamental polyhedron of a compact manifold. Explicitly, for the particular case of negative curvature, the volume of each tetrahedron in which the fundamental polyhedron can be divided is

$$v = \int_0^{\hat{C}\hat{B}\hat{D}} \frac{d\varphi}{2} \times \left\{ \frac{\operatorname{arctanh} \left[\tanh z \sec \varphi \sqrt{\cos^2 \varphi + \tan^2 \hat{B}\hat{A}\hat{C}} \right]}{\sec \varphi \sqrt{\cos^2 \varphi + \tan^2 \hat{B}\hat{A}\hat{C}}} - z \right\} \quad (19)$$

where again $z = d_{AB}$; alternatively,

$$v = \int_0^{d_{AB}} \frac{dz}{2} \times \left\{ \frac{\operatorname{arctanh} \left[\tan \hat{C}\hat{B}\hat{D} \left(\cot^2 \hat{B}\hat{A}\hat{C} \operatorname{csch}^2 z - 1 \right)^{-1/2} \right]}{\sqrt{\cot^2 \hat{B}\hat{A}\hat{C} \operatorname{csch}^2 z - 1}} \right\}. \quad (20)$$

These results must be compared with the more traditional ones given in [4], [5] and [6].

ACKNOWLEDGMENTS

The author wants to thank the anonymous people of the Brazilian state of São Paulo, who gave him financial support through grant n. 96/0052-3 of the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP); the author is also deeply grateful for the help of R.G. Teixeira and Prof. Helio V. Fagundes.

APPENDIX A: TRIGONOMETRIC IDENTITIES

In the non-euclidean geometry the trigonometric identities valid for a triangle XYZ , with right angle Z , of sides x , y and hypotenuse z are [4]

$$\sin Y = \frac{\sin \sqrt{k}y}{\sin \sqrt{k}z}; \quad \cos Y = \frac{\tan \sqrt{k}x}{\tan \sqrt{k}z}; \quad \tan Y = \frac{\tan \sqrt{k}y}{\sin \sqrt{k}x}. \quad (A1)$$

Using the second identity in the right-angled triangle BCD , of right angle C , and the third one in the right-angled triangle ABC , of right angle B , one can write, for the tetrahedron $ABCD$ built as in the section 2,

$$\cos \hat{C}\hat{B}\hat{D} = \cos \varphi = \frac{\tan \sqrt{k}d_{BC}}{\tan \sqrt{k}\rho} = \frac{\tan \hat{B}\hat{A}\hat{C} \sin \sqrt{k}d_{AB}}{\tan \sqrt{k}\rho} \quad (A2)$$

from where one obtains equation (11), after identification of d_{AB} with z .

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